

MAGNETIC LAPLACIANS OF LOCALLY EXACT FORMS ON THE SIERPINSKI GASKET

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ABSTRACT. We give a mathematically rigorous construction of a magnetic Schrödinger operator corresponding to a field with flux through finitely many holes of the Sierpinski Gasket. The operator is shown to have discrete spectrum accumulating at ∞ , and it is shown that the asymptotic distribution of eigenvalues is the same as that for the Laplacian. Most eigenfunctions may be computed using gauge transformations corresponding to the magnetic field and the remainder of the spectrum may be approximated to arbitrary precision by using a sequence of approximations by magnetic operators on finite graphs.

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1. INTRODUCTION

The properties of quantum electron on a fractal substrate and under the influence of a magnetic field were studied long ago in the physics literature [14, 6, 29, 7, 28, 17] as part of a more general program involving quasiperiodic media [32, 9], but until recently there has been no mathematically rigorous model for even formulating a magnetic Schrödinger equation on a self-similar fractal set. We remedy this in the special case of the Sierpinski Gasket with certain simple magnetic fields using mathematical developments from the study of diffusions and Laplacian-type operators on fractals using probability and functional analysis (see [8, 24, 34] and references therein) and the recent introduction of differential forms associated to this structure [11, 21, 1, 10, 20, 18, 19]. These developments in analysis on fractals have benefited from and contributed to the understanding of quantum and statistical physics [4, 5, 3, 15, 2].

Our goal in this paper is to introduce a mathematically rigorous Schrödinger equation for a magnetic operator on the Sierpinski Gasket (SG), following the

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methods of [21, 20, 18, 19], and study its spectrum, which by [19] is discrete and accumulates only at ∞ (Theorem 3.1). For reasons of mathematical simplicity we consider a somewhat unphysical situation in which the magnetic field has non-zero flux through only finitely many of the “holes” in the gasket. In this situation we are able to prove that the magnetic operator may be approximated in an appropriate sense by a renormalized sequence of magnetic operators on approximating graphs (Theorems 3.8 and 3.10). This approximation generalizes the well-known approximation of a Dirichlet form on SG by renormalized graph Dirichlet forms [22, 24]. The approximating magnetic operators provide a method for numerical study of the spectrum and some data of this type is in Section 4. Guided by the observations in this data and using the description of the Laplacian spectrum from the spectral decimation method [30, 16, 27] we show that a field through only finitely many holes of SG modifies only those eigenvalues for which the eigenfunctions have support enclosing these holes (Theorems 4.1 and 4.3), and conclude that the spectral asymptotics of the magnetic operator are the same as those of the Laplacian (Corollary 4.4). In principle, for any magnetic field of this type, one can use our methods to compute the bulk of the spectrum and the associated eigenfunctions by applying suitable gauge transformations to Laplacian eigenfunctions. For the small (asymptotically vanishing) portion of the spectrum that is not found by this method one can choose λ and compute all eigenvalues of size less than λ by solving finitely many linear algebra problems. We also give a description of the basic modification that a magnetic field makes to the Laplacian spectrum by examining periodic functions on a covering space (Section 5). In the case of SG the relevant covering space is a fractafold (as defined in [33]) called the Sierpinski Ladder [35].

2. ANALYSIS AND 1-FORMS ON SG

The Sierpinski Gasket (SG) is the attractor of the Iterated Function System $\{F_j = \frac{1}{2}(x - p_j) + p_j\}$, $j = 0, 1, 2$ for $\{p_j\}$ the vertices of an equilateral triangle in \mathbb{R}^2 . The image of SG under an m -fold composition of these maps is called an m -cell. We index these by words: let $w = w_1 w_2 \dots w_m \in \{0, 1, 2\}^m$ be a word of length $|w| = m$ and $F_w = F_{w_1} \circ \dots \circ F_{w_m}$. Then $F_w(\text{SG})$ is an m -cell. From the cellular structure of SG we obtain a sequence of graphs. Let $V_0 = \{p_0, p_1, p_2\}$ and inductively $V_m = \cup_{j=0,1,2} F_j(V_{m-1})$. The m^{th} -level graph approximation of SG is the graph with vertices V_m and edges between pairs of vertices that are contained in a common m -cell. We write $x \sim_m y$ to denote that there is an edge between $x, y \in V_m$ in the m -scale graph. The set $V_* = \cup_m V_m$ is dense in SG.

Analysis on SG is based on the existence of a Dirichlet form and an associated Laplacian. Of the available constructions [8, 26, 24] we follow the method of Kigami [24], some features of which are as follows. Proofs of all of the results stated may be found in [24, 34]. We endow SG with the (unique) self-similar probability measure μ that is invariant under the symmetries of the triangle with vertices the points p_j .

- A1 There is a Dirichlet form \mathcal{E} on SG with domain $\mathcal{F} \subset L^2(\mu)$ consisting of continuous functions. \mathcal{E} may be localized to any m -cell and is self-similar with scaling factor $\frac{5}{3}$. Specifically, for a word w with $|w| = m$ let $\mathcal{E}_w(f, g) = (\frac{5}{3})^m \mathcal{E}(f \circ F_w, g \circ F_w)$ so \mathcal{E}_w is a Dirichlet form on $F_w(\text{SG})$. Then $\mathcal{E}(f, g) = \sum_{|w|=m} \mathcal{E}_w(f, g)$.

- A2 \mathcal{E} may be obtained as a limit of forms on the graphs. For $f, g : V_* \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, let $\mathcal{E}_m(f, g) = \sum_{x \sim_m y} (f(x) - f(y))(g(x) - g(y))$. Then $(\frac{5}{3})^m \mathcal{E}_m(f, f)$ is non-decreasing and converges to $\mathcal{E}(f, f)$ if $f \in \mathcal{F}$.
- A3 From standard considerations there is a non-positive definite self-adjoint Laplacian associated to \mathcal{E} . We define $f \in \text{dom}(\Delta) \subsetneq \mathcal{F}$ to mean there is a continuous Δf such that $\mathcal{E}(f, g) = \langle -\Delta f, g \rangle_{L^2}$ for all $g \in \mathcal{F}_0$, where $\mathcal{F}_0 \subset \mathcal{F}$ is the subspace of functions that vanish on V_0 .
- A4 Let $df(p_j) = \lim_{m \rightarrow \infty} (\frac{5}{3})^m (2f(p_j) - f(F_j^{\circ m} p_{(j+1)}) - f(F_j^{\circ m} p_{(j+2)}))$, where the subscripts are taken modulo 3. If $f \in \text{dom}(\Delta)$ then this limit exists on V_0 and there is a Gauss-Green formula $\mathcal{E}(f, g) = \langle -\Delta f, g \rangle_{L^2} + \sum_{j=0}^2 df(p_j)g(p_j)$; we call $df(p_j)$ the normal derivative of f at p_j . Both df and the Gauss-Green formula may be localized to any m -cell.
- A5 Δ may be obtained as a limit of graph Laplacians. For $f : V_* \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, let $\Delta_m f(x) = \sum_{y \sim_m x} (f(y) - f(x))$. Then $\Delta f(x) = \frac{3}{2} \lim_{m \rightarrow \infty} 5^m \Delta_m f(x)$.
- A6 If $X \subset \text{SG}$ is finite and $g : X \rightarrow \mathbb{R}$ then there is a unique $f \in \mathcal{F}$ such that $f|_X = g$ and $\mathcal{E}(f)$ is minimized; f is called the harmonic extension of g and satisfies $\Delta f(x) = 0$ for $x \in \text{SG} \setminus X$. If $X = V_m$ then also $\Delta_n f(x) = 0$ for all $n > m$ and $x \in V_n \setminus V_m$, and f is called m -harmonic.

Differential forms on certain spaces that include the Sierpinski Gasket have been studied in [11, 21, 1, 10, 20, 18]. We follow the approach in [21], which introduces 1-forms as a Hilbert space \mathcal{H} generated by tensor products $f \otimes g$ with $f, g \in \mathcal{F}$, and which is a module over \mathcal{F} . There is then a derivation $\partial : \mathcal{F} \rightarrow \mathcal{H}$ such that $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f, f)$ and the image of ∂ is the space of exact forms.

The key feature that we need from [21] is that the action of \mathcal{F} on \mathcal{H} by multiplication extends to permit multiplication by much more general functions. In particular, multiplication by the characteristic function $\mathbf{1}_w$ of an m -cell $F_w(X)$ is well-defined. This permits a cellular decomposition of \mathcal{H} akin to that described in (A1) and a notion of graph approximation like that in (A2). Proofs of the following results are in [21].

- F1 Let \mathcal{H}_w be the space of 1-forms constructed from $(\mathcal{E}_w, \mathcal{F}|_{F_w(\text{SG})})$ in the same manner as \mathcal{H} is constructed from $(\mathcal{E}, \mathcal{F})$. If $h_w = f|_{F_w(\text{SG})} \otimes g|_{F_w(\text{SG})}$ then the map $h_w \mapsto (f \circ F_w) \otimes (g \circ F_w) = h$ takes the dense subspace of generators of \mathcal{H}_w to those of \mathcal{H} and has $\|h_w\|_{\mathcal{H}_w}^2 = (\frac{5}{3})^m \|h \circ F_w\|_{\mathcal{H}}^2$, so extends to an isomorphism of \mathcal{H}_w to \mathcal{H} .
- F2 \mathcal{H}_w is isometrically isomorphic to the subspace $\{a\mathbf{1}_w : a \in \mathcal{H}\}$ via the continuous extension of the identification of $f|_{F_w(\text{SG})} \otimes g|_{F_w(\text{SG})}$ with $(f \otimes g)\mathbf{1}_w$ and there is a direct sum decomposition $\mathcal{H} = \bigoplus_{|w|=m} \mathcal{H}_w$.
- F3 Let \mathcal{H}_m be the subspace of \mathcal{H} generated by $\{f \otimes \mathbf{1}_w : f \text{ is } m\text{-harmonic and } |w| = m\}$. Then $\mathcal{H}_m \subset \mathcal{H}_{m+1}$ for all m and $\cup_m \mathcal{H}_m$ is dense in \mathcal{H} . The preceding results imply that \mathcal{H}_m is isomorphic to a direct sum of copies of \mathcal{H}_0 , with one copy of \mathcal{H}_0 for each m -cell. Moreover \mathcal{H}_0 is isomorphic to the harmonic functions modulo constants on SG, and is obtained from this space by applying the derivation ∂ .

Though it is not made explicit in [21], the result in (F3) gives a connection to 1-forms on graphs. Recall that a 1-form on a graph is simply a function on the set of directed edges. Let $a \in \mathcal{H}_m$ and e_{xy} denote the edge from x to y in the m -scale graph. Take w with $|w| = m$ so $F_w(\text{SG})$ is the unique cell containing e_{xy} and use (F3) to obtain a harmonic function modulo constants A_w corresponding to $a\mathbf{1}_w$. If we set $A(e_{xy}) = A_w(y) - A_w(x)$ then A is a well-defined function on directed

edges, so is a 1-form on the m -scale graph. Moreover it is exact at scale m because on each m -cell $F_w(\text{SG})$ it is the derivative of A_w . The norm of $a \in \mathcal{H}_m$ is simply $\|a\|_{\mathcal{H}}^2 = \sum_{|w|=m} \mathcal{E}(A_w) = \sum_{x \sim_m y} A(e_{xy})^2$.

This permits us to understand the space \mathcal{H} as a generalization of $(\mathcal{E}, \mathcal{F})$, because it exhibits the \mathcal{H} -norm as a renormalized limit of L^2 -norms. To make this connection more precise we need some definitions. Let h_j denote the harmonic function on SG which has values $h_j(p_j) = 0$, $h_j(p_{j-1}) = -1$ and $h_j(p_{j+1}) = 1$.

Definition 2.1. For any two points joined by an edge in the m -scale graph there is $j \in \{0, 1, 2\}$ and a word w with $|w| = m$ such that the points are $x = F_w(p_{j-1})$ and $y = F_w(p_{j+1})$ (subindices are taken modulo 3). Define $\text{Tr}_m : \mathcal{H} \rightarrow \mathcal{H}_m$ by setting the value on the edge e_{xy} from x to y to be

$$(\text{Tr}_m a)(e_{xy}) = \frac{1}{3} \langle a, \partial h_j \rangle_{\mathcal{H}}.$$

A sequence $\{a_m\}_1^\infty \subset \mathcal{H}$ is called *compatible* if $\text{Tr}_m a_{m+1} = a_m$ for all m .

The following theorem should be compared to the results in Section 4 of [1]. It gives a full description of 1-forms on SG as limits of 1-forms on the approximating graphs.

Theorem 2.2. *The map $\text{Tr}_m : \mathcal{H} \rightarrow \mathcal{H}_m$ is a projection. If $a \in \mathcal{H}$ then the sequence $\{a_m\}$ of projections onto \mathcal{H}_m is compatible, $a_m \rightarrow a$ in \mathcal{H} and $\|a_m\|_{\mathcal{H}} \uparrow \|a\|_{\mathcal{H}}$. Conversely, if $\{a_m\}$ is a compatible sequence then $a_m \in \mathcal{H}_m$ for all m ; if we further assume that $\|a_m\|_{\mathcal{H}}$ is bounded then there is $a \in \mathcal{H}$ such that $a_m \rightarrow a$ and a_m is the projection of a to \mathcal{H}_m for all m .*

Proof. The main thing we need to prove is that Tr_m is the projection onto \mathcal{H}_m . From (F2) it is apparent that the projection can be taken one cell at a time, and the self-similarity in (F1) implies that all cells are the same, so it suffices to show Tr_0 is the projection onto \mathcal{H}_0 . We recall that \mathcal{H}_0 is obtained from the 2-dimensional space of harmonic functions on SG by applying the derivation.

Let \tilde{h}_j be harmonic on SG with $\tilde{h}_j(p_j) = 1$, $\tilde{h}_j(p_{j+1}) = \tilde{h}_j(p_{j-1}) = 0$. Symmetry shows that h_j and \tilde{h}_j are orthogonal, so ∂h_j and $\partial \tilde{h}_j$ are an orthogonal basis for \mathcal{H}_0 . Suppose we project $a \in \mathcal{H}$ onto $a_0 \in \mathcal{H}_0$ and compute the corresponding function A_\emptyset . Since $\tilde{h}_j(p_{j+1}) = \tilde{h}_j(p_{j-1})$, the difference $A(p_{j+1}) - A(p_j)$ is determined by the component involving h_j . Precisely, it is

$$A(p_{j+1}) - A(p_j) = \frac{1}{\mathcal{E}(h_j)} \langle a, \partial h_j \rangle_{\mathcal{H}} (h(p_{j+1}) - h(p_j)) = \frac{2}{6} \langle a, \partial h_j \rangle_{\mathcal{H}} = \text{Tr}_0 a(e_{p_{j-1}p_j}).$$

Thus the trace assigns the same values to the edges as does the projection, and they must coincide. Note that, in particular, this means the values of $\text{Tr}_0 a$ on the three edges e_{01} , e_{12} , e_{20} must sum to zero, and indeed we find from the definition that they do because $\sum_j h_j$ is identically zero, so $\sum_j \text{Tr}_0 a(e_{j(j+1)}) = 0$.

Having established that Tr_m is the projection onto \mathcal{H}_m it is immediate that the sequence of projections a_m of $a \in \mathcal{H}$ is compatible, and (F3) shows $a_m \rightarrow a$ in \mathcal{H} and $\|a_m\|_{\mathcal{H}} \uparrow \|a\|_{\mathcal{H}}$. For the converse, if a_m is a compatible sequence then the fact that $a_m = \text{Tr}_m a_{m+1}$ implies $a_m \in \mathcal{H}_m$ for all m and that $\|a_m\|_{\mathcal{H}}$ is an increasing sequence. If we suppose that $\|a_m\|_{\mathcal{H}}$ is bounded then using the Pythagorean decomposition $\|a_n\|_{\mathcal{H}}^2 = \|a_m\|_{\mathcal{H}}^2 + \|a_n - a_m\|_{\mathcal{H}}^2$, $n > m$, for projection in a Hilbert space we see $\|a_n - a_m\|_{\mathcal{H}}^2 \leq (\sup_n \|a_n\|_{\mathcal{H}}^2) - \|a_m\|_{\mathcal{H}}^2 \rightarrow 0$ as $m, n \rightarrow \infty$, so the sequence is Cauchy

with limit $a \in \mathcal{H}$. Finally, the composition $\text{Tr}_m \circ \text{Tr}_{m+1} \circ \cdots \circ \text{Tr}_n$ shows $a_m = \text{Tr}_m a_n$ for all $n > m$ and, by taking the limit, $a_m = \text{Tr}_m a$. \square

3. MAGNETIC FORMS, MAGNETIC LAPLACIAN AND GAUGE TRANSFORMATIONS

Following [18, 19] a magnetic differential may be defined as a deformation of ∂ . To do so we treat a real-valued 1-form $a \in \mathcal{H}$ as an operator $\mathcal{F} \rightarrow \mathcal{H}$ via multiplication, so $f \mapsto fa$. Then $(\partial + ia) : \mathcal{F} \rightarrow \mathcal{H}$ is the magnetic differential obtained by deforming ∂ via the form $a \in \mathcal{H}$. With this approach an essential result is the following theorem.

Theorem 3.1 ([19]). *The quadratic form $\mathcal{E}^a(f) = \|(\partial + ia)f\|_{\mathcal{H}}^2$ with domain \mathcal{F} is closed on $L^2(\mu)$. Thus there is an associated non-positive definite self-adjoint magnetic (Neumann) Laplacian \mathcal{M}_N^a satisfying*

$$\mathcal{E}^a(f, g) = \langle -\mathcal{M}_N^a f, g \rangle_{L^2(\mu)}$$

for all $g \in \mathcal{F}$. Moreover \mathcal{M}_N^a has compact resolvent, so the spectrum of $-\mathcal{M}_N^a$ is a sequence $0 \leq \kappa_1 \leq \kappa_2 \leq \cdots$ accumulating only at ∞ .

The same argument provides that the quadratic form $(\mathcal{E}^a, \mathcal{F}_0)$ is closed on the space $L^2(\text{SG} \setminus V_0, \mu)$ and defines a magnetic (Dirichlet) Laplacian \mathcal{M}_D^a with compact resolvent and $\mathcal{E}^a(f, g) = \langle -\mathcal{M}_D^a f, g \rangle$ for all $g \in \mathcal{F}_0$. Henceforth we will just use the Dirichlet magnetic operator and will denote it \mathcal{M}^a . Much of our work transfers to the Neumann magnetic operator with minor changes.

Remark 3.2. We are using the complexification of each of the spaces $L^2(\mu)$, \mathcal{F} , \mathcal{H} , $\text{dom}(\Delta)$ as well as the subspaces \mathcal{F}_0 , \mathcal{H}_m , etc. These are standard, but for the convenience of the reader we recall that one may complexify \mathcal{F} by endowing $\mathcal{F} + i\mathcal{F}$ with the form

$$\mathcal{E}(f, g) = \mathcal{E}(f_1, g_1) - i\mathcal{E}(f_1, g_2) + i\mathcal{E}(f_2, g_1) + \mathcal{E}(f_2, g_2)$$

where $f = f_1 + if_2$ and $g = g_1 + ig_2$. In this case the finite approximations in (A.2) become $\mathcal{E}_m(f, g) = \sum_{x \sim_m} (f(x) - f(y))(\overline{g(x) - g(y)})$. One may then construct \mathcal{H} from the complexified version of $\mathcal{F} \otimes \mathcal{F}$ in the same manner as was done in the real case in [21] and discussed in Section 2.

We wish to study the spectrum of \mathcal{M}^a by making graph approximations. For this reason we introduce a graph magnetic form and a graph magnetic Laplacian. The connection between these and \mathcal{E}^a and \mathcal{M}^a is not immediately obvious but will rapidly become apparent.

Definition 3.3. Suppose $a \in \mathcal{H}$ is real-valued and for each $m \in \mathbb{N}$ let a_m be the projection of a to \mathcal{H}_m . For $f, g : V_* \rightarrow \mathbb{C}$ define

$$(3.1) \quad \mathcal{E}_m^{a_m}(f) = \sum_{x, y : x \sim_m y} \left| f(x) - f(y) e^{ia_m(e_{xy})} \right|^2$$

$$(3.2) \quad \mathcal{M}_m^{a_m} f(x) = - \sum_{y : y \sim_m x} \left(f(x) - f(y) e^{ia_m(e_{xy})} \right) \quad \text{for } x \in V_m \setminus V_0.$$

We have the usual relation

$$(3.3) \quad \mathcal{E}_m^{a_m}(f, g) = \langle -\mathcal{M}_m^{a_m} f, g \rangle_{l^2(V_m)},$$

when $g = 0$ on V_0 , as may be verified by direct computation:

$$\begin{aligned}
& 2 \sum_{x,y:x \sim_m y} \left(f(x) - f(y)e^{ia_m(e_{xy})} \right) \overline{\left(g(x) - g(y)e^{ia_m(e_{xy})} \right)} \\
&= \sum_{x \in V_m \setminus V_0} \overline{g(x)} \sum_{y \sim_m x} \left(f(x) - f(y)e^{ia_m(e_{xy})} \right) - \sum_{y \in V_m \setminus V_0} \overline{g(y)} \sum_{x \sim_m y} \left(f(x)e^{-ia_m(e_{xy})} - f(y) \right) \\
&= \sum_{x \in V_m \setminus V_0} \overline{g(x)} \sum_{y \sim_m x} \left(f(x) - f(y)e^{ia_m(e_{xy})} \right) + \sum_{x \in V_m \setminus V_0} \overline{g(x)} \sum_{y \sim_m x} \left(f(x) - f(y)e^{ia_m(e_{yx})} \right) \\
&= 2 \sum_{x \in V_m \setminus V_0} (-\mathcal{M}_m^{a_m} f(x)) \overline{g(x)}.
\end{aligned}$$

Note that we need not sum over V_0 because g vanishes there. The equality holds for arbitrary g if $\sum_{y \sim_m x} (f(x) - f(y)e^{ia_m(e_{xy})}) = 0$ for $x \in V_0$.

Lemma 3.4. $\left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f)$ converges as $m \rightarrow \infty$ if and only if $f \in \mathcal{F}$.

Proof. Observe from $||f(x)| - |f(y)|| \leq |f(x) - f(y)e^{ia_m(e_{xy})}|$ that the convergence in the statement implies $\left(\frac{5}{3}\right)^m \mathcal{E}_m(|f|)$ is finite and therefore $|f| \in \mathcal{F}$. In particular f is bounded. The converse assumption $f \in \mathcal{F}$ also ensures f is bounded.

Using boundedness of f we may estimate as follows

$$\begin{aligned}
\left| f(x) - f(y)e^{ia_m(e_{xy})} \right|^2 &\leq \left(|f(x) - f(y)| + |f(y)| |1 - e^{ia_m(e_{xy})}| \right)^2 \\
&\leq 2|f(x) - f(y)|^2 + 2\|f\|_\infty^2 |a_m(e_{xy})|^2
\end{aligned}$$

and similarly

$$|f(x) - f(y)|^2 \leq \left| f(x) - f(y)e^{ia_m(e_{xy})} \right|^2 + 2\|f\|_\infty^2 |a_m(e_{xy})|^2.$$

As previously discussed, $\left(\frac{5}{3}\right)^m \sum_{x \sim_m y} |a_m(e_{xy})|^2 = \|a_m\|_{\mathcal{H}}^2 \leq \|a\|_{\mathcal{H}}^2$, so convergence of $\left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f)$ is equivalent to convergence of $\left(\frac{5}{3}\right)^m \mathcal{E}_m(f)$ and thus to $f \in \mathcal{F}$. \square

Of course one should expect that $\left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f)$ converges to $\mathcal{E}^a(f)$, but we have only proved this under a condition akin to assuming $a \in \mathcal{H}$ is locally exact. Note that in the classical (Euclidean) setting all 1-forms are locally exact because the space is locally topologically trivial, but this is not the case on fractals.

Definition 3.5. A 1-form $a \in \mathcal{H}$ is called exact if there is $A \in \mathcal{F}$ such that $\partial A = a$. It is locally exact if there is an open cover such that it is exact on the open sets. Equivalently, it is locally exact if there is a finite partition of $\text{SG} = \cup_j X_{w_j}$ of SG into cells $X_{w_j} = F_{w_j}(\text{SG})$ such that a is exact on each cell, meaning there are $A_{w_j} \in \mathcal{F}$ so $a \mathbb{1}_{w_j} = (\partial A_{w_j}) \mathbb{1}_{w_j}$ for all j . We say a is exact at scale m if this is the smallest integer for which the partition can be chosen to consist of m -cells.

It is proved in [19] that when a is real-valued and exact there is a Coulomb gauge transformation which conjugates \mathcal{E}^a to \mathcal{E} and \mathcal{M}^a to Δ . Specifically, one has from Corollary 5.6 of [19]

$$(3.4) \quad \mathcal{E}^a(f) = \mathcal{E}(e^{iA} f)$$

$$(3.5) \quad \mathcal{M}^a f = e^{-iA} \Delta(e^{iA} f)$$

In fact rather more can be obtained from the discussion at the end of Section 5 of [19], using the notion of a Coulomb gauge.

Definition 3.6. Suppose $a \in \mathcal{H}$ is real-valued. We say a admits a Coulomb gauge if there is $e^{iA} \in \mathcal{F}$ such that $e^{-iA}\partial(e^{iA}) = a$, and a admits a local Coulomb gauge if this is true on the cells of a finite partition.

Remark 3.7. If a admits a Coulomb gauge then it is locally exact, because e^{iA} is uniformly continuous and thus has a logarithm in \mathcal{F} on all sufficiently small cells. However, having a Coulomb gauge is weaker than (global) exactness because it is possible to have $e^{iA} \in \mathcal{F}$ with A locally but not globally in \mathcal{F} . To see the distinction, suppose that a is locally exact with $a = \partial A_j$ on cells X_{w_j} . The A_{w_j} are defined up to additive constants, and a is exact if and only if we can choose these constants so $A = A_{w_j}$ on X_{w_j} is continuous on SG. By contrast, a has a Coulomb gauge if we can choose the constants so that $e^{iA} = e^{iA_{w_j}}$ on X_{w_j} is continuous on SG, so in this latter case we may permit jump discontinuities that are integer multiples of 2π at intersection points of the cells.

From Theorem 5.9 of [19] both (3.4) and (3.5) are valid when a admits a Coulomb gauge. Note that this Corollary relies on the hypothesis that for connected open sets U , $\partial f \mathbf{1}_U = 0$ implies f is constant on U . This is valid on SG because we can write U as a connected union of cells, whence at any finite scale the cellular decomposition of $\|\cdot\|_{\mathcal{H}}$ allows us to assume the restriction of ∂f to each cell is zero. Since each cell is self-similar to SG it suffices to note that if $\mathcal{E}(f) = \|\partial f\|_{\mathcal{H}}^2 = 0$ then f is constant by the properties of resistance forms.

It should be noted that when there is a Coulomb gauge we may immediately write a gauge transformation of $\mathcal{E}_m^{a_m}$ and $\mathcal{M}_m^{a_m}$, because in this case the function $e^{iA} \in \mathcal{F}$ has an m -harmonic approximation (see (A6 for the definition). The m -harmonic approximation has the same values as e^{iA} at points of V_m , so denoting it with $(e^{iA})_m$ we can use to write

$$e^{ia_m(e_{xy})} = (e^{iA(y)})_m (e^{-iA(x)})_m$$

for all $x \sim_m y$ in V_m , and therefore

$$(3.6) \quad \mathcal{E}_m^{a_m}(f) = \sum_{x,y:x \sim_m y} \left| f(x)(e^{iA(x)})_m - f(y)(e^{iA(y)})_m \right|^2 = \mathcal{E}_m(e^{iA}f),$$

and similarly, for $x \in V_m$,

$$(3.7) \quad \mathcal{M}_m^{a_m} f(x) = -(e^{-iA(x)})_m \sum_{y:y \sim_m x} \left(f(x)(e^{iA(x)})_m - f(y)(e^{iA(y)})_m \right)$$

$$(3.8) \quad = e^{-iA} \Delta_m (e^{iA} f).$$

For forms that admit a local Coulomb gauge our graph magnetic energies converge to the magnetic energy on the fractal.

Theorem 3.8. *If $a \in \mathcal{H}$ is real-valued and has a local Coulomb gauge at scale n then*

$$\left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f) \rightarrow \mathcal{E}^a(f) \text{ as } m \rightarrow \infty.$$

Proof. By hypothesis we may partition SG as $\cup_{|w|=n} F_w(\text{SG})$ and have functions $e^{iA_w} \in \mathcal{F}$ such that

$$\mathcal{E}^a(f) = \sum_{|w|=n} \mathcal{E}_{X_w}(e^{iA_w} f|_{X_w})$$

where \mathcal{E}_{X_w} is the Dirichlet form on the cell $X_w = F_w(\text{SG})$, so is just a rescaling of the global Dirichlet form.

On each cell the m -scale energy \mathcal{E}_m converges to \mathcal{E} , so take $m > n$ sufficiently large that

$$\mathcal{E}_{X_w,m}(e^{iA_w}f|_{X_w}) \leq \mathcal{E}_{X_w}(e^{iA_w}f|_{X_w}) \leq \frac{\epsilon}{N} + \mathcal{E}_{X_w,m}(e^{iA_w}f|_{X_w}).$$

where N is the number of n -cells. Now by (3.6) each of the $\mathcal{E}_{X_w,m}(e^{iA_w}f|_{X_w})$ is that part of the sum for $\mathcal{E}_m^{a_m}(f)$ which corresponds to the edges in X_w , so summing over the finite collection of cells in the truncated sum gives $\mathcal{E}_m^{a_m}(f)$ and we have shown it is within 2ϵ of $\mathcal{E}^a(f)$. \square

Remark 3.9. We conjecture that Theorem 3.8 holds without the restriction that a admits a local Coulomb gauge. Note, however, that we will also need the Coulomb gauge restriction to prove our results on the spectrum of \mathcal{M}^a in Section 4, so little is lost by making this assumption here too.

Theorem 3.10. *Suppose $a \in \mathcal{H}$ is real-valued and has a local Coulomb gauge at scale n . Then $f \in \text{dom}(\mathcal{M}^a)$ if and only if $\frac{3}{2}5^m \mathcal{M}_m^{a_m} f$ converges uniformly on $V_* \setminus V_0$ to a continuous function Φ . In this case the continuous extension of Φ to SG is $\mathcal{M}^a f$.*

Proof. First assume the uniform convergence to a continuous Φ . For any $g \in \mathcal{F}$ that vanishes on V_0 define functions h_m which are harmonic at scale m and have values

$$h_m(x) = \frac{3}{2}5^m (\mathcal{M}_m^{a_m} f(x)) \overline{g(x)} \quad \text{for } x \in V_m \setminus V_0.$$

Obviously $h_m(x)$ converges uniformly on SG to the continuous extension of $\Phi(x)\overline{g(x)}$. What is more, the integral of the m -harmonic function which is 1 at $x \in V_m \setminus V_0$ and zero at all other points of V_m is $\frac{2}{3}3^{-m}$ so we may compute

$$\int h_m(x) d\mu = \left(\frac{5}{3}\right)^m \sum_{x \in V_m} (\mathcal{M}_m^{a_m} f(x)) \overline{g(x)}$$

Then (3.3) says that

$$\int h_m(x) d\mu = -\left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f, g)$$

By Theorem 3.8 and the parallelogram law the right side converges to $-\mathcal{E}^a(f, g)$, and since the left side converges to $\int \Phi \bar{g} d\mu = \langle \Phi, g \rangle_{L^2(\mu)}$ and $g \in \mathcal{F}_0$ is arbitrary it must be that $f \in \text{dom}(\mathcal{M}^a)$ with $\mathcal{M}^a f$ being the continuous extension of Φ to SG .

Conversely we have $\mathcal{E}^a(f, g) = -\langle \mathcal{M}^a f, g \rangle_{L^2(\mu)}$ for all $g \in \mathcal{F}_0$ and will make a careful choice of g . Fix $x \in V_* \setminus V_0$ and $m \geq n$. Since a has a Coulomb gauge at scale n we may find $e^{iA_w}, e^{iA_{w'}} \in \mathcal{F}$ so that $a\mathbb{1}_w = e^{-iA_w} \partial(e^{iA_w})\mathbb{1}_w$ and $a\mathbb{1}_{w'} = e^{-iA_{w'}} \partial(e^{iA_{w'}})\mathbb{1}_{w'}$, where $F_w(\text{SG})$ and $F_{w'}(\text{SG})$ are the two n -cells that meet at x . However the e^{iA_w} and $e^{iA_{w'}}$ are defined only up to multiplicative constants constants of norm 1, so we can arrange that they join continuously at x and write both as e^{iA} . Now let ϕ_m be the m -harmonic function which is equal $e^{iA(x)}$ at x and zero at all other points of V_m and define $\psi_m = e^{-iA}\phi_m$. Note that both ϕ_m and ψ_m are identically zero off $F_w(\text{SG}) \cup F_{w'}(\text{SG})$, so the behavior of A off this set does not affect ψ_m . Since ψ_m is a product of elements of \mathcal{F} and is zero at V_0 it is in \mathcal{F}_0 . Using this and the fact that ψ_m is supported on the set where the gauge transformation is valid

$$-\langle \mathcal{M}^a f, \psi_m \rangle_{L^2(\mu)} = \mathcal{E}^a(f, \psi_m) = \mathcal{E}(e^{iA}f, e^{iA}\psi_m) = \mathcal{E}(e^{iA}f, \phi_m)$$

but ϕ_m is m -harmonic, so

$$\mathcal{E}(e^{iA}f, \phi_m) = \left(\frac{5}{3}\right)^m \mathcal{E}_m(e^{iA}f, \phi_m) = \left(\frac{5}{3}\right)^m \mathcal{E}_m^{a_m}(f, \psi_m)$$

and inserting (3.3) we have only the terms involving x , so

$$3^m \langle \mathcal{M}^a f, \psi_m \rangle_{L^2(\mu)} = 5^m \mathcal{M}_m^{a_m} f(x).$$

We assumed $\mathcal{M}^a f$ was continuous, and it is obvious the support of the ψ_m converges to x , so the proof will be complete if we show $3^m \int \psi_m d\mu \rightarrow \frac{2}{3}$. However e^{iA} is continuous, so its restriction to the support of ψ_m converges uniformly to $e^{iA(x)}$ as $m \rightarrow \infty$. If χ_m denotes the harmonic function which is 1 at x and zero on the other points of V_m then we conclude $\psi_m - \chi_m$ converges uniformly to zero. Moreover $3^m \int \chi_m d\mu = \frac{2}{3}$ for all m by elementary symmetry considerations, so the proof is complete. \square

Theorem 3.11. *Suppose $a \in \mathcal{H}$ is real-valued and admits a local Coulomb gauge at scale n . If $f \in \text{dom}(\mathcal{M}^a)$, then the magnetic normal derivative*

$$d^a f(p) = \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m p} (f(p) - e^{i(A_p(x) - A_p(p))} f(x))$$

exists at each $p \in V_0$ and for $g \in \mathcal{F}$ we have the Gauss-Green formula

$$(3.9) \quad \mathcal{E}^a(f, g) = -\langle \mathcal{M}^a f, g \rangle_{L^2(\mu)} + \sum_{x \in V_0} (d^a f(p)) \overline{g(p)}.$$

If, in addition, there is $A_p \in \mathcal{F}$ such that $\partial A_p = a$ on a neighborhood of p , and the usual normal derivative $dA_p(p)$ exists, then $df(p)$ exists and

$$(3.10) \quad d^a f(p) = e^{-iA_p(p)} d(f e^{iA_p}) = df(p) + i f(p) dA_p(p)$$

Proof. Fix $g \in \mathcal{F}$. For each m and each $p \in V_0$ use the construction of ψ_m from the proof of Theorem 3.10 to obtain a function ψ_m^p which is 1 at p , zero at all other points of V_m and such that if e^{iA_p} is the local Coulomb gauge at p then $e^{iA_p} \psi_m^p$ is m -harmonic. Let $g_m = \sum_{p \in V_0} g(p) \psi_m^p$. Then $g - g_m \in \mathcal{F}_0$ and therefore $\mathcal{E}^a(f, g - g_m) = -\langle \mathcal{M}^a f, g - g_m \rangle_{L^2(\mu)}$. Since $\mathcal{M}^a f$ is continuous and $g - g_m \rightarrow g$ in $L^2(\mu)$ we find that $\mathcal{E}^a(f, g_m)$ converges. For large enough m the gauge transform and the definition of ψ_m^p imply

$$\begin{aligned} \mathcal{E}^a(f, g_m) &= \sum_p \overline{g(p)} \mathcal{E}(e^{iA_p} f, e^{iA_p} \psi_m^p) \\ &= \left(\frac{5}{3}\right)^m \sum_p \overline{g(p)} \mathcal{E}_m(e^{iA_p} f, e^{iA_p} \psi_m^p) \\ &= \left(\frac{5}{3}\right)^m \sum_p \overline{g(p)} \sum_{x \sim_m p} (f(p) - e^{i(A_p(x) - A_p(p))} f(x)) \end{aligned}$$

so that the magnetic normal derivative exists and (3.9) holds.

When dA_p exists we have $A_p(x) - A_p(p) = -\left(\frac{3}{5}\right)^m dA_p(p) + o\left(\left(\frac{3}{5}\right)^m\right)$ for both $x \sim_m p$. Thus we compute

$$\begin{aligned} df(p) &= \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m p} (f(p) - f(x)) \\ &= \lim_{m \rightarrow \infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m p} \left((f(p) - f(x) e^{i(A_p(x) - A_p(p))}) + f(x) (e^{i(A_p(x) - A_p(p))} - 1) \right) \end{aligned}$$

$$= d^a f(p) - if(p)dA_p(p)$$

which gives the second conclusion of the theorem. \square

It is apparent that we can localize the magnetic Gauss-Green formula to any cell. Doing so allows us to give necessary and sufficient conditions for defining a function in $\text{dom}(\mathcal{M}^a)$ piecewise.

Theorem 3.12. *Suppose $a \in \mathcal{H}$ is real-valued and admits a local Coulomb gauge. Let $X_1 = F_{w_1}(SG)$ and $X_2 = F_{w_2}(SG)$ be two cells with $X_1 \cap X_2 = \{x\}$ and assume we have functions f_j and u_j from $\mathcal{F}|_{X_j}$ such that $\mathcal{M}^a f_j = u_j$, $j = 1, 2$. In order that the piecewise functions $f = f_j$ on X_j and $u = u_j$ on X_j for $j = 1, 2$ satisfy $\mathcal{M}^a f = u$ it is necessary and sufficient that both are continuous, $f_1(x) = f_2(x)$ and $u_1(x) = u_2(x)$, and also that $d^a f_1(x) + d^a f_2(x) = 0$.*

Proof. The role of the continuity assumption is elementary, so we focus on the condition on d^a . By localizing (3.9) to X_1 and X_2 we may write the hypothesis $\mathcal{M}^a u_j = f_j$ as

$$(3.11) \quad \mathcal{E}_{X_j}^a(f_j, g) = -\langle u_j, g \rangle_{L^2(\mu, X_j)} + \sum_{p \in V_0} (d^a f_j(F_{w_j}(p))) \overline{g(F_{w_j}(p))}$$

for $j = 1, 2$. Similarly, $\mathcal{M}^a u = f$ on the union means that for functions g which vanish on $(F_{w_1}(V_0) \cup F_{w_2}(V_0)) \setminus \{x\}$ we have

$$\mathcal{E}_{X_1 \cup X_2}^a(f, g) = -\langle u, g \rangle_{L^2(\mu, X_1 \cup X_2)}.$$

Comparing this to the sum of (3.11) for $j = 1, 2$ we see that they are the same if and only if all the terms from the sums over V_0 vanish. Our hypothesis on g ensures these sums only contain the two terms at x , so the quantity which must vanish is $(d^a f_1(x) + d^a f_2(x))\overline{g(x)}$, and $g(x)$ can take any value. \square

We conclude this section with a discussion of the structure of the subspace of exact forms on SG and its complementary subspace in \mathcal{H} . Recall that the exact forms are the image of the map $\partial : \mathcal{F} \rightarrow \mathcal{H}$. Since $\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f)$ and \mathcal{F} modulo constants is a Hilbert space, the exact 1-forms are a complete, hence closed, subspace of \mathcal{H} . We write P for the projection onto the exact forms and P^\perp for the orthogonal projection. It is proven in [21] that $P\mathcal{H}_m$ is the space obtained by applying ∂ to the m -harmonic functions, while $P^\perp\mathcal{H}_m$ is the space of m -harmonic 1-forms. A 1-form is m -harmonic if on each m -cell $X_w = F_w(SG)$ it is $(\partial h_w)\mathbb{1}_w$ for some m -harmonic function h_w , and for any point $x \in V_m$ the sum of the normal derivatives $\sum_w dh_w(x)$ over the cells meeting at x is zero.

The self-similarity of the space \mathcal{H}_m ensures we may understand the structure of \mathcal{H}_m by studying the structure of \mathcal{H}_1 . This is generated by the harmonic functions modulo constants on the 1-cells. It is convenient to incorporate the condition on constants by assuming the harmonic functions have mean zero, so the sum of the values at points $F_j(V_0)$ is zero for each $j \in \{0, 1, 2\}$. One can then check that $P\mathcal{H}_1$ is 5-dimensional. In fact, the 5-dimensional space generated by 1-harmonic functions that are mean-zero on SG can be made mean-zero on each $F_j(SG)$ by subtracting an appropriate mean-zero function that is harmonic on all of SG, so this space decomposes into the 2-dimensional space $\mathcal{H}_0 = P\mathcal{H}_0$ and a 3-dimensional complement. The remaining space, $P^\perp\mathcal{H}_1$ is 1-dimensional and corresponds to a loop around the central hole. We let $b \in \mathcal{H}_1$ be the element with counterclockwise orientation shown in Figure 1(a), multiplied by $1/\sqrt{30}$ so that $\|b\|_{\mathcal{H}} = 1$. It is also

convenient to choose harmonic functions on the 1-cells as shown in Figure 1(b) such that applying ∂ gives $\sqrt{30}b$. Although this latter is not a function on SG it is a function B on the disjoint union $\sqcup_{j=0,1,2} F_j(SG)$.

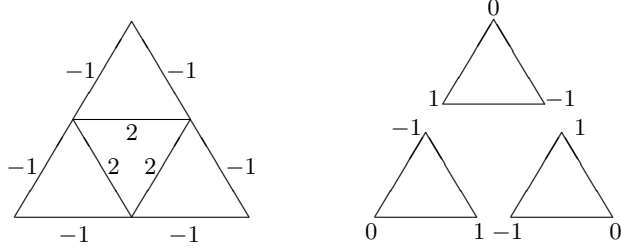


FIGURE 1. (a) The 1-form $\sqrt{30}b$, with orientation clockwise around each 1-cell, hence counterclockwise around the central hole, and (b) The harmonic function B on disjoint 1-cells.

It is apparent that the set $\{b \circ F_w\}$ of 1-forms span the space of harmonic forms $P^\perp \mathcal{H}$. If $b \circ F_w$ and $b \circ F_{w'}$ are from disjoint cells then the direct sum decomposition in (F2) implies they are orthogonal, and by computing $\text{Tr}_0 b = 0$ from the formula in Definition 2.1 we find $b \circ F_w$ and $b \circ F_{w'}$ are orthogonal if $|w| \neq |w'|$. Thus $\{b \circ F_w\}$ is an orthogonal basis for $P^\perp \mathcal{H}$, and for real values β_w

$$(3.12) \quad \left\| \sum_{m=1}^{\infty} \sum_{|w|=m} \beta_w b \circ F_w \right\|_{\mathcal{H}}^2 = \sum_{m=1}^{\infty} \left(\frac{5}{3}\right)^m \sum_{|w|=m} \beta_w^2$$

if the latter series converges. Moreover, $a \in \mathcal{H}$ is locally exact if and only if $P^\perp a$ is a series of this type with only finitely many terms, it has Coulomb gauge if and only if all β_w in the series for $P^\perp a$ are integer multiples of 2π , and it has scale n Coulomb gauge if and only if all β_w in the series for $P^\perp a$ which have $|w| > n$ are integer multiples of 2π . Note that the final point and (3.12) gives another proof that every form admitting a local Coulomb gauge is locally exact, though not necessarily at the same scale.

4. SPECTRA OF MAGNETIC OPERATORS WITH LOCAL COULOMB GAUGE

In this section we study the spectrum of Dirichlet magnetic operators \mathcal{M}^a , which we know from Theorem 3.1 is pure point. Our approach relies heavily on the spectral decimation property of the Laplacian on SG [30, 31, 16] and associated properties of the eigenfunctions [12, 23]. Spectral decimation says that if f is an eigenfunction of Δ on SG then there is m_0 (called the generation of birth) and a sequence $\{\lambda_m\}_{m_0}^{\infty}$ such that $\Delta_m f = \lambda_m f$ for all $m \geq m_0$. The sequence $\{\lambda_m\}$ is related to the eigenvalue by $\lambda_m(5 - \lambda_m) = \lambda_{m-1}$ and $\frac{3}{2} \lim 5^m \lambda_m = \lambda$. One way to view this graph eigenfunction equation is as follows: if on each m -cell $F_w(SG)$ we have f_w such that $\Delta f_w = \lambda f_w$ then defining f piecewise to be f_w on $F_w(SG)$ we have $\Delta f = \lambda f$ if and only if f is continuous and $\Delta_m f = \lambda_m f$. Comparing this to the usual gluing property we see that the discrete eigenfunction equation encodes that the normal derivatives sum to zero at the points of V_m . The equivalence of these conditions may also be verified using the explicit formulas for the normal derivatives from [13].

We wish to study the spectrum of \mathcal{M}^a via the finite approximations $\mathcal{M}_m^{a_m}$, so in light of the results of the previous section it makes sense to only consider real-valued $a \in \mathcal{H}$ which admit a local Coulomb gauge at scale n . By the discussion following Definition 3.5 we may also assume that $Pa = 0$, because we can gauge transform to remove this part of a . Doing so will not change the eigenvalues of \mathcal{M}^a and will simply conjugate the eigenfunctions. Under these assumptions let $m \geq n$ and e^{iA_w} be the gauge transform on the m -cell $F_w(\text{SG})$. Then u_w satisfies $\mathcal{M}_m^{a_m} u_w = \lambda u_w$ on $F_w(\text{SG})$ if and only if $f_w = e^{iA_w} u_w$ and $\Delta f_w = \lambda f_w$ on the cell. The condition for gluing the u_w into a piecewise defined eigenfunction with $\mathcal{M}^a u = \lambda u$ is that they join continuously and $\sum_w d^a u_w(p) = 0$, where the sum is over the cells meeting at $p \in V_m$. From $Pa = 0$ we have $\sum_w dA_w(p) = 0$ for all p , so by (3.10) our condition becomes $\sum_w e^{-iA_w(p)} df_w(p) = \sum_w du_w = \sum_w d^a u_w = 0$. But $\Delta e^{-iA_w(p)} f_w(x) = \lambda e^{-iA_w(p)} f_w(x)$, so the normal derivatives sum to zero at p if and only if $\Delta_m e^{-iA_w(p)} f_w(p) = \lambda_m e^{-iA_w(p)} f_w(p)$, which is precisely $\mathcal{M}_m^{a_m} u_w = \lambda_m u_w$. Thus we can study $\mathcal{M}^a u = \lambda u$ by examining $\mathcal{M}_m^{a_m} u = \lambda_m u$ for $m \geq n$.

As described at the end of the previous section, the assumptions we have on a imply that there are real numbers β_w with $\beta_w \in 2\pi\mathbb{Z}$ for $|w| > n$, such that

$$(4.1) \quad a = \sum_{m=1}^{\infty} \sum_{|w|=m} \beta_w b \circ F_w,$$

$$\|a\|_{\mathcal{H}}^2 = \sum_{m=1}^{\infty} \left(\frac{5}{3}\right)^m \sum_{|w|=m} \beta_w^2 < \infty.$$

Since all terms in this expression are self-similar it is clear that a significant step is to understand the spectrum of $\mathcal{M}^{\beta b}$, in which case we can look at $\mathcal{M}_1^{\beta b}$.

The results of some numerical investigations into the spectrum of \mathcal{M}^b are shown in Figure 2. One can see the structure of the spectrum inherited from the spectral decimation process, which copies and expands the spectrum with each level of approximation.

Of particular note is the existence of many eigenvalues that do not vary with β , and are therefore independent of the field. These can be seen in Figure 2 as horizontal lines. This pattern persists for more complicated magnetic operators \mathcal{M}^a with local Coulomb gauge: when m is sufficiently large we find that $\mathcal{M}_m^{a_m}$ has a large number of eigenvalues that are the same as those of Δ_m . This turns out to be a straightforward consequence of the structure of the eigenfunctions of the Laplacian.

Theorem 4.1. *Suppose f is an eigenfunction of Δ with eigenvalue λ and the support of f is a finite union of cells $\cup X_k$ on which a has a Coulomb gauge, so there is $e^{iA} \in \mathcal{F}$ such that $e^{-iA}(\partial e^{iA})\mathbb{1}_{\cup X_k} = a\mathbb{1}_{\cup X_k}$. Then fe^{-iA} is an eigenfunction of \mathcal{M}^a with eigenvalue λ .*

Proof. This is a direct computation from the validity of the gauge transformation on $\cup X_k$, because for $g \in \mathcal{F}_0$

$$\mathcal{E}^a(fe^{-iA}, g) = \mathcal{E}(f, e^{iA}g) = -\lambda \langle f, e^{iA}g \rangle = -\lambda \langle fe^{-iA}, g \rangle. \quad \square$$

Remark 4.2. This result can also be thought of in terms of the gluing result in Theorem 3.12. By construction fe^{-iA} satisfies the eigenfunction equation for \mathcal{M}^a on $\cup X_k$. From the fact that f is identically zero outside $\cup X_k$ we see that df must be zero at the boundary points of $\cup X_k$. Using (3.10) with $f = df = 0$ on the boundary

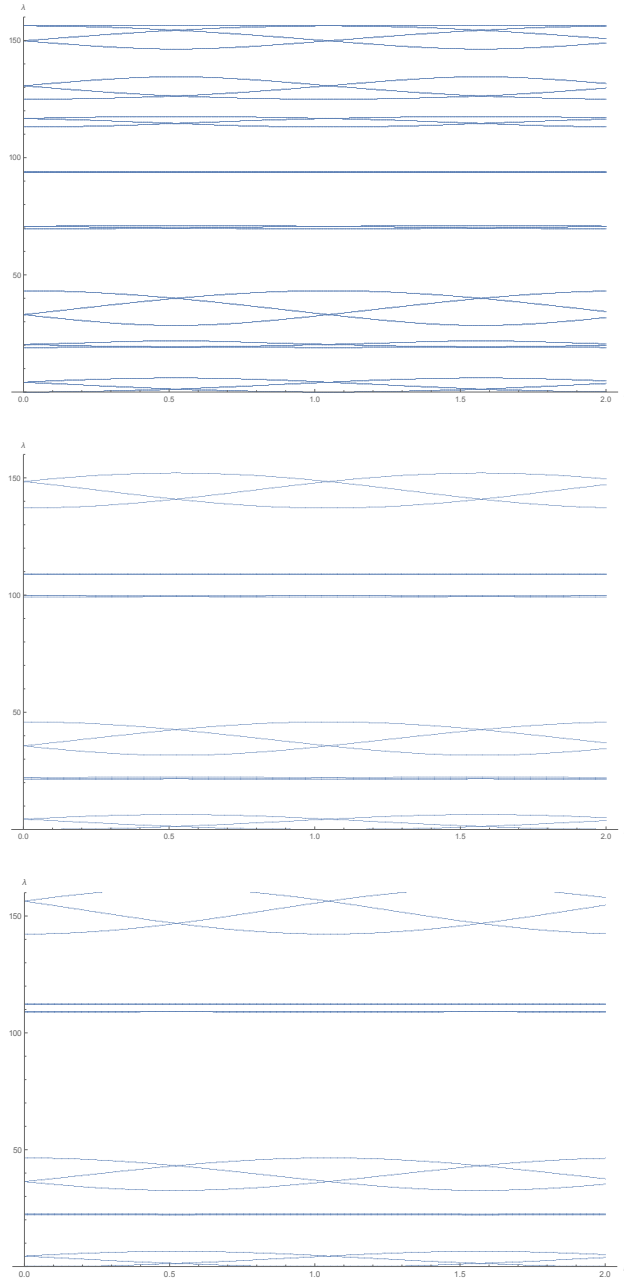


FIGURE 2. Eigenvalues less than 160 and $0 \leq \beta \leq 2$ for the (from top to bottom) 4th, 5th, and 6th level approximation to $\mathcal{M}^{\beta b}$

of $\cup X_k$ we have $fe^{-iA} = d^a(fe^{-iA}) = 0$ there also, so extending fe^{-iA} by zero gives a smooth solution of the eigenfunction equation on SG.

In order to see why this result determines many eigenfunctions of \mathcal{M}^a we need some more consequences of the spectral decimation method, particularly those

from [12, 23]. Our presentation of them follows the elementary exposition in [34], except that our bases for the 5-series eigenspaces are more like those in [12]. In order to describe these bases we define a chain of m -cells to be a sequence $X_k = F_{w_k}(\text{SG})$, $k = 1, \dots, K$ such that $|w_k| = m$ for all k and $X_k \cap X_{k+1} = \{x_k\}$ is a sequence of $K - 1$ distinct points from V_m . We say the chain is simple if $X_k \cap X_{k'} = \emptyset$ unless $|k - k'| \leq 1$.

- (S1) For a Dirichlet eigenvalue λ of Δ on SG with eigenfunction f , let $m(\lambda)$ be its generation of birth and λ_m be the spectral decimation sequence, so $\Delta_m f = \lambda_m f$, $\lambda_m(5 - \lambda_m) = \lambda_{m-1}$ and $\frac{3}{2} \lim 5^m \lambda_m = \lambda$. Then $\lambda_{m(\lambda_0)} \in \{2, 5, 6\}$ and $\lambda_m \notin \{2, 5, 6\}$ for $m > m(\lambda)$. We let $\sigma_s = \{\lambda : \lambda_{m(\lambda_0)} = s\}$ for $s = 2, 5, 6$, and call these the 2, 5, and 6 series eigenfunctions.
- (S2) From the preceding, $\lambda_m = \frac{1}{2}(5 \pm \sqrt{25 - 4\lambda_{m-1}}) = \Phi_{\pm}(\lambda_{m-1})$. For convergence of $5^m \lambda_m$ the positive root can occur at most finitely often, so there is $m_1(\lambda)$ called the generation of fixation such that $\lambda_m = \Phi_{-}(\lambda_{m-1})$ for all $m > m_1$. Writing $\Phi_{-}^{\circ m}$ for the m -fold composition, the function $\mathcal{R}(\tau) = \lim_m 5^m \Phi_{-}^{\circ m}(\tau)$ is analytic, $\mathcal{R}(0) = 0$ and $\mathcal{R}'(0) \neq 0$. Knowing the generation of fixation the eigenvalue is $\lambda = 5^{m_1} \mathcal{R}(\lambda_{m_1})$.
- (S3) If $\lambda \in \sigma_2$ then $m(\lambda) = 1$, its eigenspace is 1-dimensional, and the eigenfunctions are fully symmetric under the dihedral symmetry group of the triangle.
- (S4) If $\lambda \in \sigma_5$ then $m(\lambda) \geq 1$. All eigenfunctions vanish on $V_{m(\lambda)-1}$ and the eigenspace has dimension $\frac{1}{2}(3^{m(\lambda)-1} + 3)$. There is a basis for the 5-series eigenfunctions in which each is supported on a simple chain of $(m(\lambda) - 1)$ -cells in which X_{k_1} and X_{k_K} contain distinct points of V_0 .
- (S5) If $\lambda \in \sigma_6$ then $m(\lambda) \geq 2$. The eigenspace has dimension $\frac{1}{2}(3^{m(\lambda)} - 3)$, and there is a basis in which each eigenfunction is supported on the union of two $(m(\lambda) - 1)$ -cells meeting at a point of $V_{m(\lambda)-1} \setminus V_0$.

A small comment about the 5-series basis is in order. With generation of birth $m + 1$ there is a function supported on an m -cell with the following property: given an m -chain with both ends on V_0 there is an arrangement of copies of the function along the cells in the chain such that the resulting function extends smoothly by 0 to give a 5-series eigenfunction on SG. This arrangement is unique up to multiplying the eigenfunction by a scalar. In [12] a basis is given in which each eigenfunction is supported on an m -cell chain from p_0 to either p_1 or p_2 , but the chains given are not simple. In particular it follows from [12] that the number of m -cell chains between two points of V_0 is $\frac{1}{2}(3^{m-1} + 1)$. Observe that each simple m -chain determines an $(m - 1)$ -chain by taking the parent cells of the m -cells in the chain. Conversely an $(m - 1)$ -chain determines a simple m -chain by taking, in each $(m - 1)$ -cell X_k , the two m -cells which form the shortest m -cell chain from x_{k-1} to x_k . From this bijection between simple m -chains and $(m - 1)$ -chains we see that the number of simple m -cell chains between two specified points of V_0 is $\frac{1}{2}(3^{m-2} + 1)$, and therefore the number of such chains joining pairs of points from V_0 is $\frac{1}{2}(3^{m-1} + 3)$, which is the dimension of a 5-series eigenspace with generation of birth m . Moreover it is easy to prove inductively that the eigenfunctions corresponding to these chains are linearly independent. When $m = 2$ this can be done by hand (as was done in [12]). For the inductive step observe that if a linear combination of eigenfunctions corresponding to simple m -chains is zero then it is zero on each cell $F_j(X)$, $j = 0, 1, 2$. Then precomposing the piece on $F_j(X)$ with F_j^{-1} gives a vanishing linear combination of

eigenfunctions corresponding to $(m-1)$ -chains, and these are linearly independent by the inductive hypothesis.

Theorem 4.3. *If a is a real-valued form with local Coulomb gauge at scale n and λ is a Laplacian eigenvalue with generation of birth $m(\lambda) > n$ then λ is also an eigenvalue of \mathcal{M}^a , and the corresponding eigenfunction is obtained from the Laplacian eigenfunction by a gauge transformation.*

Proof. If a is as described then on every n -cell $F_w(\text{SG})$ we have a gauge function e^{iA_w} , which is determined up to a multiplicative constant. For λ as described the Laplacian eigenfunction is supported either on simple chain of $(m(\lambda)-1) \geq n$ cells, or on the union of two $(m(\lambda)-1)$ cells, which we denote $X_k = F_{w_k}(\text{SG})$. In either case simplicity of the chain ensures we may choose the values $e^{iA_w(x_k)}$, where $x_k = X_k \cap X_{k+1}$, so that $e^{iA} = e^{iA_{w_k}}$ on X_k is continuous, hence a Coulomb gauge on $\cup X_k$. The result then follows from Theorem 4.1. \square

Corollary 4.4. *If a is a real-valued form with local Coulomb gauge at scale n then \mathcal{M}^a has the same spectral asymptotics as Δ . Specifically, let $\rho^a(x)$ be the counting function of \mathcal{M}^a , so $\rho^a(x) = \#\{\lambda \in \sigma_D : \lambda \leq x\}$. There is a non-constant periodic function χ of period $\log 5$ such that*

$$\lim_{x \rightarrow \infty} \rho^a(x) x^{-\log 3 / \log 5} - \chi(\log x) = 0.$$

The function χ is independent of a , so is the same as that occurring for the Laplacian spectrum.

Proof. For $a = 0$ this is simply the spectral asymptotic for the Laplacian, and follows from a more general analysis in [25]. When $a \neq 0$ the result follows from the fact that eigenvalues with generation of birth less than n make an asymptotically small contribution to the spectrum. To make this precise we reason as follows.

The eigenvalues and eigenfunctions of \mathcal{M}^m obey spectral decimation for all sufficiently large m , so for each eigenvalue λ there is a sequence λ_m , $m \geq m_0$ as in (S1) and the eigenvalue is determined at the generation of fixation as described in (S2). Following this line of reasoning, for a specified x there is m_1 comparable to $\log x$ such that all eigenvalues $\lambda \leq x$ are of the form $5^{m_1} \Phi(\lambda_{m_1})$ for λ_{m_1} an eigenvalue of $\mathcal{M}_{m_1}^{a_{m_1}}$. Hence it suffices to know what proportion of the eigenvalues of $\mathcal{M}_{m_1}^{a_{m_1}}$ have generation of birth $\leq n$. At each m the number of newly born eigenvalues is comparable to 3^m , and these split according to the positive and negative roots in the spectral decimation to give a multiple of $2^{m_1-m} 3^m$ eigenvalues at the generation of fixation, so the number of eigenvalues born before n but fixed at m_1 is comparable to 3^n , while the total number fixed at m_1 is comparable to 3^{m_1} . Thus the proportion of eigenvalues of \mathcal{M}^a that differ from those of Δ and are less than x is bounded by a multiple of $3^n/x$ for large x , and goes to zero as $x \rightarrow \infty$. \square

Theorem 4.3 also gives all of the spectrum of $e^{i\beta b}$ except that born at generation 1. We can get the rest by direct computation. If we label the points of $V_1 \setminus V_0$ as q_j , $j = 0, 1, 2$, then symmetry suggests we ought to have eigenfunctions f_k of $\mathcal{M}_1^{a_1}$ with values on V_1 given by $f_k(q_j) = e^{ijk2\pi/3}$. Indeed

$$\mathcal{M}_1^{\beta b} f_k = \left(4 - 2 \cos\left(\frac{2\pi k}{3} + \frac{2\beta}{\sqrt{30}}\right)\right) f_k$$

from which we can determine the eigenfunctions by applying 5Φ . Ideally we would like to be able to use this information to compute the bottom of the spectrum

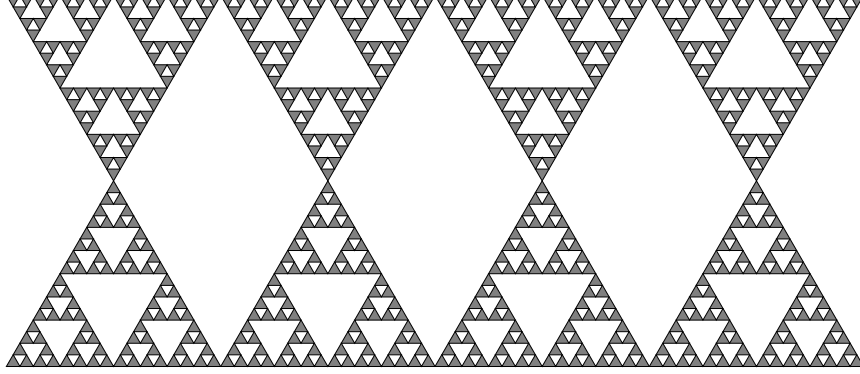


FIGURE 3. The Ladder Fractafold

for \mathcal{M}^a in the case where a is given by (4.1), at least in some special cases, but unfortunately we do not know how to do this.

5. SPECTRUM OF $\mathcal{M}^{\beta b}$ VIA THE LADDER FRACTAFOLD

An alternative approach to the problem of determining the spectrum of $\mathcal{M}^{\beta b}$ is to lift the problem to a periodic version on a suitable covering space using a technique from [35]. To avoid merely repeating the results of the previous section we illustrate this method by computing the spectrum of the Neumann magnetic operator.

The space we use is called the Ladder Fractafold based on the Sierpinski gasket, and is denoted LF. [35] gives a general method for analyzing the spectrum of a fractafold constructed by gluing copies of SG arranged according to a graph. For LF, let the vertices of a graph Γ_0 be three copies of \mathbb{Z} , labelled $\{x_{k+\frac{1}{2}}\}$, $\{w_k\}$, and $\{y_{k+\frac{1}{2}}\}$ and the edges be such that $w_k, x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}$ is a complete graph on 3 vertices, and so is $w_k, y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}$. Then LF is obtained by replacing these complete 3-graphs with copies of SG, see Figure 3.

According to the analysis in [35], the spectrum of LF can be determined from the graph of the cells and their connectivity. If we label the cell with vertices $w_k, x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}$ by a_k and that with vertices $w_k, y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}$ by b_k and treat $\{a_k\} \cup \{b_k\}$ as vertices of a graph Γ with edges when the corresponding cells intersect, then Γ is a ladder as shown in Figure 4. If $-\Delta_\Gamma$ is the usual discrete Laplacian on Γ it has absolutely continuous spectrum $[0, 6]$. One can prove the resolvent is unbounded by considering two sets of functions that satisfy an eigenfunction equation but are not in L^2 : $\{\phi_\theta\}$ such that $\phi_\theta(a_k) = \phi_\theta(b_k) = e^{ik\theta}$ with eigenvalue $2 - 2\cos\theta$ (these are even in the reflection exchanging a_k and b_k), and $\{\psi_\theta\}$ such that $\psi_\theta(a_k) = -\psi_\theta(b_k) = e^{ik\theta}$ with eigenvalue $4 - 2\cos\theta$ (these are odd in the reflection exchanging a_k and b_k). In both cases $0 \leq \theta \leq \pi$. From Theorem 3.1 of [35] and their discussion in Example 5.2, this spectrum is the same as that of $-\Delta_{\Gamma_0}$. Moreover they relate the spectrum $\sigma(-\Delta_{\Gamma_0})$ to that of the Laplacian on the fractafold as follows.

Theorem 5.1 (Theorem 2.3 of [35]). *Using the function \mathcal{R} from (S2) let*

$$\Sigma_\infty = 5 \left(\mathcal{R}\{2\} \cup \bigcup_{0}^{\infty} 5^m \mathcal{R}\{3, 5\} \right),$$

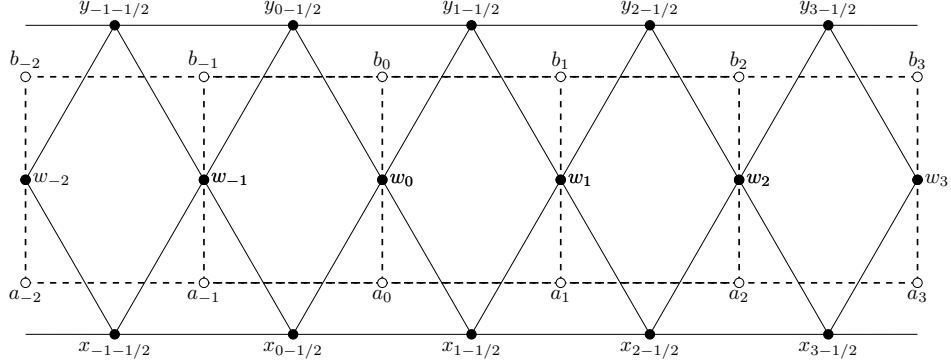


FIGURE 4. The graphs Γ_0 (unfilled vertexes and dashed edges), and Γ (filled vertexes, solid edges)

$$\Sigma'_\infty = 5 \left(\bigcup_{m=0}^{\infty} 5^m \mathcal{R}\{3, 5\} \right) \subset \Sigma_\infty.$$

Then for Δ the Laplacian on the fractafold obtained by gluing according to Γ_0

$$\mathcal{R}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma'_\infty \subset \sigma(-\Delta_{LF}) \subset \mathcal{R}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma_\infty.$$

To connect this to the study of the magnetic operator $\mathcal{M}^{\beta b}$ we “fold” the ladder along the center-line parallel to its length, so the point $x_{k+\frac{1}{2}}$ is identified with $y_{k+\frac{1}{2}}$ for all k , and obtain a fractafold is in Figure 5, which we call the folded ladder fractafold, or FLF. The FLF is a covering space for SG in which the loop around the central hole of the V_1 graph has been trivialized. The covering map takes each cell a_k in the fractafold to a 1-cell of SG in a 3-periodic manner, identifying a_k with the cell $F_{k \bmod 3}(SG)$, w_k with $p_{k \bmod 3} \in V_0$ and mapping both $x_{k+\frac{1}{2}}$ and $y_{k+\frac{1}{2}}$ to the same point of $V_1 \setminus V_0$. We arrange the map so that the line through the $x_{k+\frac{1}{2}}$ wraps in a counterclockwise direction around the central hole in the V_1 graph as k increases.

Lemma 5.2. *There is a bijection taking each Neumann eigenfunction f of Δ on SG with eigenvalue λ to a solution \tilde{f} of $\Delta_{LF}\tilde{f} = \lambda\tilde{f}$ which is symmetrical under the central line reflection and is 3-periodic.*

Proof. For the definition and properties of the Laplacian on LF and FLF we refer to [35]. A function satisfying $\Delta_{FLF}\hat{f} = \lambda\hat{f}$ on FLF unfolds to give a function \tilde{f} on LF. This function satisfies $\Delta_{LF}\tilde{f} = \lambda\tilde{f}$ if and only if its normal derivatives at each w_k sum to zero; given the symmetry, this happens if and only if $d\tilde{f} = 0$ at all points w_k . At the same time, the period 3 covering of SG by FLF ensures that 3-periodic solutions of $\Delta_{FLF}\hat{f} = \lambda\hat{f}$ on FLF correspond to eigenfunctions on SG in such a way that the normal derivatives at points w_k correspond to those on V_0 . \square

Remark 5.3. We could do something similar for the Dirichlet eigenfunctions on SG by considering antisymmetry in the center line.

More importantly, the same thing happens for the magnetic operator $\mathcal{M}^{\beta b}$. The only modification required for the proof is that the symmetric unfolding of a solution of $\Delta_{FLF}\hat{f} = \lambda\hat{f}$ from FLF to LF gives a solution of $\Delta_{LF}\tilde{f} = \lambda\tilde{f}$ if and only if $d^{\beta b}\tilde{f}(w_k)$

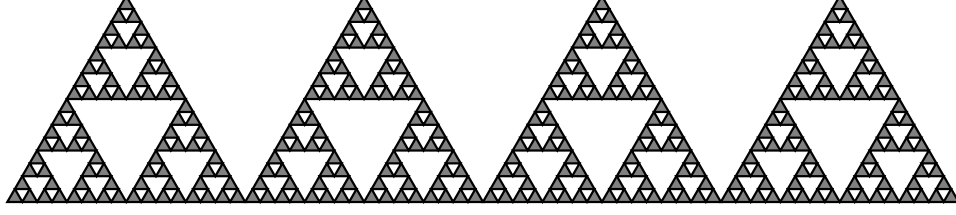


FIGURE 5. The folded Sierpinski Ladder Fractafold

sums to zero for all k . However (3.10) and the fact that $db = 0$ sums to zero at each w_k ensures the Neumann condition is still the correct one. To make this argument we need 1-forms and magnetic forms on LF and FLF; their properties are substantially similar to those on SG, and we refer to [21, 19] for more details.

Corollary 5.4. *There is a bijective map which takes each Neumann eigenfunction of $\mathcal{M}^{\beta b}$ on SG with eigenvalue λ to a solution \tilde{f} of $\mathcal{M}_{LF}^{\beta b} \tilde{f} = \lambda \tilde{f}$ that is symmetrical under the central line reflection and 3-periodic. Here $\mathcal{M}_{LF}^{\beta b}$ is the magnetic operator corresponding to the symmetric 3-periodic lift of βb to LF.*

The preceding result is significant because passing to FLF trivializes the loop where b is not exact, so we might expect βb to be exact on FLF. This is not literally true because the periodic extension of βb to FLF will not have finite energy, simply because it is periodic. However our reasoning regarding the gauge transformation is still valid: we can define $e^{i\beta B}$, which is globally continuous and locally in the domain of the Dirichlet form on FLF, such that $\mathcal{M}^{\beta b} f = e^{-i\beta B} \Delta_{FLF}(e^{i\beta B} f)$ for any f in the domain of $\mathcal{M}^{\beta b}$ with compact support, and take limits to extend this operation to L^2 .

Theorem 5.5. *The spectrum of the Neumann magnetic operator $\mathcal{M}^{\beta b}$ on SG is*

$$\sigma(\mathcal{M}^{\beta b}) = \mathcal{R}\left\{2 - 2\cos\left(\frac{2k\pi}{3} - \frac{2\beta}{\sqrt{30}}\right)\right\}_{k=0}^2 \cup \Sigma'_\infty$$

Proof. The periodic extension of βb to FLF has gauge e^{iB} where B is harmonic on each cell a_k and has values

$$B(x_{k+\frac{1}{2}}) = \frac{2\beta k}{\sqrt{30}} + \frac{1}{2}$$

and $B(w_k) = 0$ for all k . We use the same notation for the symmetric extension to LF. The gauge transformation is valid and reduces the problem to finding those elements of the spectrum of the Laplacian on LF for which the associated function is symmetric in the center line and, after application of the gauge transformation, is 3-periodic. By Theorem 5.1 and elementary arguments about the eigenfunctions associated to Σ_∞ and Σ'_∞ this includes all of Σ'_∞ but not $5\mathcal{R}\{2\}$. The remaining values correspond to spectral values from the symmetric functions ϕ_θ on Γ . According to [35] the corresponding functions on LF are equal to $e^{i(k+\frac{1}{2})\theta}$ at $x_{k+\frac{1}{2}}$ and $e^{ik\theta}$ at w_k . When multiplied by the gauge, these have

$$e^{iB} \phi_\theta(x_{k+\frac{1}{2}}) = \exp i\left((k + \frac{1}{2})\theta + \frac{2\beta k}{\sqrt{30}} + \frac{1}{2}\right)$$

which is periodic of period 3 in k if and only if $3\theta + \frac{6\beta}{\sqrt{30}} \equiv 0 \pmod{2\pi}$. Using this to determine θ , the fact that the eigenvalue on Γ was $2 - 2\cos\theta$ and the preceding reasoning from Theorem 5.1 completes the proof. \square

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